MA 441, Fall 2011

Assignment 7.

This homework is due *Tuesday* Nov 1.

There are total 42 points in this assignment. 38 points is considered 100%. If you go over 38 points, you will get over 100% for this homework and it will count towards your course grade.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper *and give credit to your collaborators in your pledge*. Your solutions should contain full proofs. Bare answers will not earn you much.

This assignment covers sections 3.5–3.7 and beginning of 4.1 in Bartle–Sherbert.

(1) [2pt] (3.4.1) Give example of a bounded sequence that is not a Cauchy sequence.

(2) [3pt] (3.4.2a) Show directly from the definition that the sequence $\left(\frac{n+1}{n}\right)$ is a Cauchy sequence.

(3) [4pt] (3.5.9) If $0 < r < 1$ and $|x_{n+1} - x_n| < r^n$ for all $n \in \mathbb{N}$, show that (x_n) is a Cauchy sequence.

(4) [3pt] Let $X = (x_n)$ be a sequence in $\mathbb{R}$. Is it true that if for any $\varepsilon > 0$, there is a natural number $H = H(\varepsilon)$ such that $|x_H - x_{H+1}| < \varepsilon$, then X is a Cauchy sequence? (*Hint*: inspect partial sums of harmonic series, or look at the exercise 3.5.5 in Textbook).

(5) [3pt] (3.7.2) Let $p \in \mathbb{N}$. Show that the series $\sum\limits_{n=0}^{\infty} a_n$ converges if and only if the series $\sum\limits_{n=p}^{\infty} a_n$ converges. (The value of the sum may be different, of course.) Conclude that the convergence of a sequence is not affected by changing a finite number of terms.

(6) [3pt] (3.7.4) If $\sum x_n$ and $\sum y_n$ are convergent, show directly from the definition that $\sum(x_n + y_n)$ is convergent.

(7) [3pt] Give an example of divergent series $\sum x_n$ and $\sum y_n$, such that $\sum(x_n + y_n)$ is convergent. Alternatively, instead of giving an explicit example, explain why it exists.

(8) [3pt] Do there exist a convergent series $\sum x_n$ and a divergent series $\sum y_n$ such that $\sum(x_n + y_n)$ is convergent?

— see next page —

(9) (Theorem 4.1.2) Let $A \subseteq \mathbb{R}$. Prove that

(a) [3pt] If a number $c \in \mathbb{R}$ is a cluster point of A, then there exists a sequence (a_n) in A such that $\lim(a_n) = c$ and $a_n \neq c$ for all $n \in \mathbb{N}$.

(b) [3pt] If there exists a sequence (a_n) in A such that $\lim(a_n) = c$ and $a_n \neq c$ for all $n \in \mathbb{N}$, then c is a cluster point of A.

(10) (Modified 4.1.1) In each case below, find a number $\delta > 0$ such that the corresponding inequality holds for all x such that $0 < |x - c| < \delta$. Give a *specific number* as your answer, for example $\delta = 0.0001$, or $\delta = 2.5$, or $\delta = 3/14348$, etc. (Not necessarily the largest possible.)

(a) [2pt] $|x^3 - 1| < 1/2$, $c = 1$. (*Hint:* $x^3 - 1 = (x-1)(x^2 + x + 1)$.)

(b) [2pt] $|x^3 - 1| < 10^{-3}$, $c = 1$.

(c) [2pt] $|x^3 - 1| < \frac{1}{10^{-3}}$, $c = 1$.

(d) [3pt] $|x + x^2 + 1/x - 6.5| < 1/2$, $c = 2$.

(e) [3pt] $|x^2 \sin x^3 - 0| < 0.001$, $c = 0$.